DATA DEPENDENCE RESULTS OF A NEW MULTISTEP AND S-ITERATIVE SCHEMES FOR CONTRACTIVE-LIKE OPERATORS

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Abstract. In this paper, we prove that convergence of a new iteration and S-iteration can be used approximate the fixed points of contractive-like operators. We also prove some data dependence results for these new iteration and S-iteration schemes for contractive-like operators. Our results extend and improve some known results in the literature.

1. Introduction

Contractive mappings and iteration procedures are some of the main tools in the study of fixed point theory. There are many contractive mappings and iteration schemes that have been introduced and developed by several authors to serve various purposes in the literature of this highly active research area, viz., [2, 39, 28, 17, 3, 33, 38, 23, 4, 40, 10, 27], among others.

Whether an iteration method used in any investigation converges to a fixed point of a contractive type mapping corresponding to a particular iteration process is of utmost importance. Therefore it is natural to see many works related to convergence of iteration methods, such as [37, 6, 7, 29, 9, 15, 24, 20, 1, 21].

Fixed point theory is concerned with investigating a wide variety of issues such as the existence (and uniqueness) of fixed points, the construction of fixed points, etc. One of these themes is data dependency of fixed points. Data dependency of fixed points has been the subject of research in fixed point theory for some time now, and data dependence research is an important theme in its own right.

Several authors who have made contributions to the study of data dependence of fixed points are Rus and Muresan [12], Rus et al. [13, 14], Berinde [36], Espínola and Petruşel [22], Markin [16], Chifu and Petruşel [5], Olantiwo [18, 19], Şoltuz [30, 31], Şoltuz and Grosan [32], Chugh and Kumar [25] and the references therein.

This paper is organized as follows. In Section 1 we present a brief survey of some known contractive mappings and iterative schemes and collect some preliminaries that will be used in the proofs of our main results. In Section 2 we show that the convergence of a new multi-step iteration, which is a special case of Jungck multistep-SP iterative process defined in [11] and S-iteration(due to Agarwal et al.) can be used approximate the fixed points of contractive-like operators. Motivated by the works of Şoltuz [30, 31], Şoltuz and Grosan [32], and Chugh and Kumar

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[25], we prove two data dependence results for the new multi-step iteration and S-iteration schemes by employing contractive-like operators.

As a background of our exposition, we now mention some contractive mappings and iteration schemes.

In [34] Zamfirescu established an important generalization of the Banach fixed point theorem using the following contractive condition: For a mapping $T : E \to E$, there exist real numbers a, b, c satisfying $0 < a < 1$, $0 < b$, $c < 1/2$ such that, for each pair x, $y \in X$, at least one of the following is true:

$$\left\{ \begin{array}{ll} (\mathrm{z}_1) & \|Tx - Ty\| \leq a \|x - y\|, \\ (\mathrm{z}_2) & \|Tx - Ty\| \leq b (\|x - Tx\| + \|y - Ty\|), \\ (\mathrm{z}_3) & \|Tx - Ty\| \leq c (\|x - Ty\| + \|y - Tx\|). \end{array} \right. \tag{1.1}$$

A mapping T satisfying the contractive conditions (z_1), (z_2) and (z_3) in (1.1) is called a Zamfirescu operator. An operator satisfying condition (z_2) is called a *Kannan operator*, while the mapping satisfying condition (z_3) is called a *Chatterjea operator*. As shown in [37], the contractive condition (1.1) leads to

$$\left\{ \begin{array}{ll} (\mathrm{b}_1) & \|Tx - Ty\| \leq \delta \|x - y\| + 2\delta \|x - Tx\| \text{ if one use } (\mathrm{z}_2), \\ \text{and} & \\ (\mathrm{b}_2) & \|Tx - Ty\| \leq \delta \|x - y\| + 2\delta \|x - Ty\| \text{ if one use } (\mathrm{z}_3), \end{array} \right. \tag{1.2}$$

for all x, $y \in E$ where $\delta := \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}$, $\delta \in [0, 1)$, and it was shown that this class of operators is wider than the class of Zamfirescu operators. Any mapping satisfying condition (b_1) or (b_2) is called a quasi-contractive operator.

Extending the above definition, Osilike and Udomene [20] considered operators T for which there exist real numbers $L \geq 0$ and $\delta \in [0, 1)$ such that for all x, $y \in E$,

$$\|Tx - Ty\| \leq \delta \|x - y\| + L \|x - Tx\|. \tag{1.3}$$

Imoru and Olantiwo [8] gave a more general definition: The operator T is called a contractive-like operator if there exists a constant $\delta \in [0, 1)$ and a strictly increasing and continuous function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$, such that, for each $x, y \in E$,

$$\|Tx - Ty\| \leq \delta \|x - y\| + \varphi(\|x - Tx\|). \tag{1.4}$$

A map satisfying (1.4) need not have a fixed point, even if E is complete. For example, let $E = [0, \infty)$, and define T by

$$Tx = \left\{ \begin{array}{ll} 1.0, & 0 \leq x \leq 0.8, \\ 0.6, & 0.8 < x. \end{array} \right.$$

Without loss of generality we may assume that $x < y$. Then, for $0 \leq x < y \leq 0.8$ or $0.8 < x < y$, $\|Tx - Ty\| = 0$, and (1.4) automatically satisfied.

If $0 \leq x \leq 0.8 < y$, then $\|Tx - Ty\| = 0.4$.

Define φ by $\varphi(t) = Lt$ for any $L \geq 2$. Then φ is increasing, continuous, and $\varphi(0) = 0$. Also, $\|x - Tx\| = 1 - x$, so that $\varphi(\|x - Tx\|) = L(1 - x) \geq 0.2L \geq 0.4$.

Therefore

$$0.4 = \|Tx - Ty\| \leq L \|x - Tx\| \leq \delta \|x - y\| + L \|x - Tx\|$$

for any $0 \leq \delta < 1$, and (1.4) is satisfied for $0 \leq x \leq 0.8 < y$. But T has no fixed point.

However, using (1.4) it is obvious that, if T has a fixed point, then it is unique.

Throughout this paper $\mathbb{N}$ denotes the set of all nonnegative integers. Let X be a Banach space, $E \subset X$ a nonempty closed, convex subset of X, and T a self map on E. Define $F_T := \{p \in X : \ p = Tp\}$ to be the set of fixed points of T. Let $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ and $\{\beta_n^i\}_{n=0}^{\infty}$, $i = \overline{1, k-2}$, $k \geq 2$ be real sequences in $[0, 1)$ satisfying certain conditions.

In [3] Rhoades and Şoltuz introduced a multi-step iterative procedure defined by

$$\left\{\begin{array}{l} x_0 \in E, \\ y_n^{k-1} = \left(1 - \beta_n^{k-1}\right) x_n + \beta_n^{k-1} T x_n, \ k \geq 2, \\ \quad y_n^i = \left(1 - \beta_n^i\right) x_n + \beta_n^i T y_n^{i+1}, \ i = \overline{1, k-2}, \\ x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T y_n^1, \ \ n \in \mathbb{N}. \end{array}\right. \tag{1.5}$$

The sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\left\{\begin{array}{l} x_0 \in E, \\ x_{n+1} = (1 - \alpha_n) T x_n + \alpha_n T y_n, \\ \quad y_n = (1 - \beta_n) x_n + \beta_n T x_n, \ n \in \mathbb{N}, \end{array}\right. \tag{1.6}$$

is known as the S-iteration process (see [9, 26, 27]).

S.Thianwan [33] defined a two-step iteration $\{u_n\}_{n=0}^{\infty}$ by

$$\left\{\begin{array}{l} x_0 \in E, \\ x_{n+1} = (1 - \alpha_n) y_n + \alpha_n T y_n, \\ \quad y_n = (1 - \beta_n) x_n + \beta_n T x_n, \ n \in \mathbb{N}. \end{array}\right. \tag{1.7}$$

Recently Phuengrattana and Suantai [38] introduced an SP iteration method defined by

$$\left\{\begin{array}{l} x_0 \in E, \\ x_{n+1} = (1 - \alpha_n) y_n + \alpha_n T y_n, \\ \quad y_n = (1 - \beta_n) z_n + \beta_n T z_n, \\ \quad z_n = (1 - \gamma_n) x_n + \gamma_n T x_n, \ \ n \in \mathbb{N}. \end{array}\right. \tag{1.8}$$

We shall employ the following iterative process. For an arbitrary fixed order $k \geq 2$,

$$\left\{\begin{array}{l} x_0 \in E, \\ x_{n+1} = (1 - \alpha_n) y_n^1 + \alpha_n T y_n^1, \\ \quad y_n^1 = \left(1 - \beta_n^1\right) y_n^2 + \beta_n^1 T y_n^2, \\ \quad y_n^2 = \left(1 - \beta_n^2\right) y_n^3 + \beta_n^2 T y_n^3, \\ \quad \cdots \\ y_n^{k-2} = \left(1 - \beta_n^{k-2}\right) y_n^{k-1} + \beta_n^{k-2} T y_n^{k-1}, \\ y_n^{k-1} = \left(1 - \beta_n^{k-1}\right) x_n + \beta_n^{k-1} T x_n, \ n \in \mathbb{N}, \end{array}\right. \tag{1.9}$$

or, in short,

$$\left\{\begin{array}{l} x_0 \in E, \\ x_{n+1} = (1 - \alpha_n) y_n^1 + \alpha_n T y_n^1, \\ \quad y_n^i = \left(1 - \beta_n^i\right) y_n^{i+1} + \beta_n^i T y_n^{i+1}, \ i = \overline{1, k-2}, \\ y_n^{k-1} = \left(1 - \beta_n^{k-1}\right) x_n + \beta_n^{k-1} T x_n, \ \ n \in \mathbb{N}, \end{array}\right. \tag{1.10}$$

where

$$\{\alpha_n\}_{n=0}^{\infty} \subset [0, 1), \ \sum_{n=0}^{\infty} \alpha_n = \infty, \tag{1.11}$$

and

$$\left\{\beta_n^i\right\}_{n=0}^{\infty} \subset [0,1)\,,\ i = \overline{1,k-1}. \tag{1.12}$$

Remark 1. *If $\gamma_n = 0$, then SP iteration* (1.8) *reduces to the two-step iteration* (1.7)*. By taking $k = 3$ and $k = 2$ in* (1.10) *we obtain the iterations* (1.8) *and* (1.7)*, respectively.*

We shall need following definition and lemma in the sequel.

Definition 1. [35] *Let $T,\widetilde{T} : X \to X$ be two operators. We say that $\widetilde{T}$ is an approximate operator for T if, for some $\varepsilon > 0$, we have*

$$\left\|Tx - \widetilde{T}x\right\| \leq \varepsilon,$$

for all $x \in X$.

Lemma 1. [32] *Let $\{a_n\}_{n=0}^{\infty}$ be a nonnegative sequence for which one assumes that there exists an $n_0 \in \mathbb{N}$, such that, for all $n \geq n_0$,*

$$a_{n+1} \leq (1 - \mu_n)\, a_n + \mu_n \eta_n$$

is satisfied, where $\mu_n \in (0,1)$, for all $n \in \mathbb{N}$, $\sum\limits_{n=0}^{\infty} \mu_n = \infty$ and $\eta_n \geq 0$, $\forall n \in \mathbb{N}$. Then the following holds:

$$0 \leq \lim_{n\to\infty} \sup a_n \leq \lim_{n\to\infty} \sup \eta_n.$$

2. Main Results

For simplicity we use the following notation through this section.

For any iterative process, $\{x_n\}_{n=0}^{\infty}$ and $\{u_n\}_{n=0}^{\infty}$ denote iterative sequences associated to T and $\widetilde{T}$, respectively.

Theorem 1. *Let $T : E \to E$ be a map satisfying* (1.4) *with $F_T \neq \emptyset$ and $\{x_n\}_{n=0}^{\infty}$ is a sequence defined by* (1.10)*, then the sequence $\{x_n\}_{n=0}^{\infty}$ converges to the unique fixed point of T.*

Proof. The proof can be easily obtained by using argument in the proof of ([11], Theorem 3.1). □

This result allow us to prove the following theorem.

Theorem 2. *Let $T : E \to E$ be a map satisfying* (1.4) *with $F_T \neq \emptyset$ and $\widetilde{T}$ be an approximate operator of T as in the Definition 1. Let $\{x_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$ be two iterative sequences defined by* (1.10) *and with real sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\left\{\beta_n^i\right\}_{n=0}^{\infty} \subset [0,1)$ satisfying (i) $0 \leq \beta_n^i < \alpha_n \leq 1$, $i = \overline{1,k-1}$, (ii) $\sum \alpha_n = \infty$. If $p = Tp$ and $q = \widetilde{T}q$, then we have*

$$\|p - q\| \leq \frac{k\varepsilon}{1-\delta}.$$

Proof. For a given $x_0 \in E$ and $u_0 \in E$ we consider following multistep iteration for T and $\widetilde{T}$:

$$\begin{cases} x_0 \in E, \\ x_{n+1} = (1-\alpha_n)\, y_n^1 + \alpha_n T y_n^1, \\ y_n^i = \left(1-\beta_n^i\right) y_n^{i+1} + \beta_n^i T y_n^{i+1},\ i = \overline{1,k-2}, \\ y_n^{k-1} = \left(1-\beta_n^{k-1}\right) x_n + \beta_n^{k-1} T x_n,\ k \geq 2,\ n \in \mathbb{N}, \end{cases} \tag{2.1}$$

and

$$\left\{\begin{array}{r}u_0\in E,\\ u_{n+1}=(1-\alpha_n)\,v_n^1+\alpha_n\widetilde{T}v_n^1,\\ v_n^i=\left(1-\beta_n^i\right)v_n^{i+1}+\beta_n^i\widetilde{T}v_n^{i+1},\ i=\overline{1,k-2},\\ v_n^{k-1}=\left(1-\beta_n^{k-1}\right)u_n+\beta_n^{k-1}\widetilde{T}u_n,\ k\geq 2,\ n\in\mathbb{N}.\end{array}\right. \tag{2.2}$$

Then, from (1.4), (2.1) and (2.2), we have the following estimates.

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| &= \left\|(1-\alpha_n)\left(y_n^1-v_n^1\right)+\alpha_n\left(Ty_n^1-\widetilde{T}v_n^1\right)\right\|\\ &\leq (1-\alpha_n)\left\|y_n^1-v_n^1\right\|+\alpha_n\left\|Ty_n^1-\widetilde{T}v_n^1\right\|\\ &= (1-\alpha_n)\left\|y_n^1-v_n^1\right\|+\alpha_n\left\|Ty_n^1-Tv_n^1+Tv_n^1-\widetilde{T}v_n^1\right\|\\ &\leq (1-\alpha_n)\left\|y_n^1-v_n^1\right\|+\alpha_n\left\|Ty_n^1-Tv_n^1\right\|+\alpha_n\left\|Tv_n^1-\widetilde{T}v_n^1\right\|\\ &\leq (1-\alpha_n)\left\|y_n^1-v_n^1\right\|+\alpha_n\delta\left\|y_n^1-v_n^1\right\|+\alpha_n\varphi\left(\left\|y_n^1-Ty_n^1\right\|\right)+\alpha_n\varepsilon\\ &= [1-\alpha_n(1-\delta)]\left\|y_n^1-v_n^1\right\|+\alpha_n\varphi\left(\left\|y_n^1-Ty_n^1\right\|\right)+\alpha_n\varepsilon, \qquad (2.3)\end{aligned}$$

$$\begin{aligned}\left\|y_n^1-v_n^1\right\| &= \left\|\left(1-\beta_n^1\right)\left(y_n^2-v_n^2\right)+\beta_n^1\left(Ty_n^2-\widetilde{T}v_n^2\right)\right\|\\ &\leq \left(1-\beta_n^1\right)\left\|y_n^2-v_n^2\right\|+\beta_n^1\left\|Ty_n^2-\widetilde{T}v_n^2\right\|\\ &\leq \left(1-\beta_n^1\right)\left\|y_n^2-v_n^2\right\|+\beta_n^1\left\|Ty_n^2-Tv_n^2\right\|+\beta_n^1\left\|Tv_n^2-\widetilde{T}v_n^2\right\|\\ &\leq \left(1-\beta_n^1\right)\left\|y_n^2-v_n^2\right\|+\beta_n^1\delta\left\|y_n^2-v_n^2\right\|+\beta_n^1\varphi\left(\left\|y_n^2-Ty_n^2\right\|\right)+\beta_n^1\varepsilon\\ &= \left[1-\beta_n^1(1-\delta)\right]\left\|y_n^2-v_n^2\right\|+\beta_n^1\varphi\left(\left\|y_n^2-Ty_n^2\right\|\right)+\beta_n^1\varepsilon, \qquad (2.4)\end{aligned}$$

$$\begin{aligned}\left\|y_n^2-v_n^2\right\| &= \left\|\left(1-\beta_n^2\right)\left(y_n^3-v_n^3\right)+\beta_n^2\left(Ty_n^3-\widetilde{T}v_n^3\right)\right\|\\ &\leq \left(1-\beta_n^2\right)\left\|y_n^3-v_n^3\right\|+\beta_n^2\left\|Ty_n^3-\widetilde{T}v_n^3\right\|\\ &\leq \left(1-\beta_n^2\right)\left\|y_n^3-v_n^3\right\|+\beta_n^2\left\|Ty_n^3-Tv_n^3\right\|+\beta_n^2\left\|Tv_n^3-\widetilde{T}v_n^3\right\|\\ &\leq \left(1-\beta_n^2\right)\left\|y_n^3-v_n^3\right\|+\beta_n^2\delta\left\|y_n^3-v_n^3\right\|+\beta_n^2\varphi\left(\left\|y_n^3-Ty_n^3\right\|\right)+\beta_n^2\varepsilon\\ &= \left[1-\beta_n^2(1-\delta)\right]\left\|y_n^3-v_n^3\right\|+\beta_n^2\varphi\left(\left\|y_n^3-Ty_n^3\right\|\right)+\beta_n^2\varepsilon. \qquad (2.5)\end{aligned}$$

Combining (2.3), (2.4) and (2.5) we obtain

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| \leq\ & [1-\alpha_n(1-\delta)]\left[1-\beta_n^1(1-\delta)\right]\left[1-\beta_n^2(1-\delta)\right]\left\|y_n^3-v_n^3\right\|\\ &+[1-\alpha_n(1-\delta)]\left[1-\beta_n^1(1-\delta)\right]\beta_n^2\varphi\left(\left\|y_n^3-Ty_n^3\right\|\right)\\ &+[1-\alpha_n(1-\delta)]\left[1-\beta_n^1(1-\delta)\right]\beta_n^2\varepsilon\\ &+[1-\alpha_n(1-\delta)]\beta_n^1\varphi\left(\left\|y_n^2-Ty_n^2\right\|\right)\\ &+[1-\alpha_n(1-\delta)]\beta_n^1\varepsilon+\alpha_n\varphi\left(\left\|y_n^1-Ty_n^1\right\|\right)+\alpha_n\varepsilon. \qquad (2.6)\end{aligned}$$

Thus, by induction, we get

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| &\leq [1-\alpha_n(1-\delta)] \\ &\quad \left[1-\beta_n^1(1-\delta)\right]\cdots\left[1-\beta_n^{k-2}(1-\delta)\right]\left\|y_n^{k-1}-v_n^{k-1}\right\| \\ &\quad +[1-\alpha_n(1-\delta)] \\ &\quad \left[1-\beta_n^1(1-\delta)\right]\cdots\left[1-\beta_n^{k-3}(1-\delta)\right]\beta_n^{k-2}\varphi\left(\left\|y_n^{k-1}-Ty_n^{k-1}\right\|\right) \\ &\quad +\cdots+[1-\alpha_n(1-\delta)]\,\beta_n^1\varphi\left(\left\|y_n^2-Ty_n^2\right\|\right)+\alpha_n\varphi\left(\left\|y_n^1-Ty_n^1\right\|\right) \\ &\quad +[1-\alpha_n(1-\delta)]\left[1-\beta_n^1(1-\delta)\right]\cdots\left[1-\beta_n^{k-3}(1-\delta)\right]\beta_n^{k-2}\varepsilon \\ &\quad +\cdots+[1-\alpha_n(1-\delta)]\,\beta_n^1\varepsilon+\alpha_n\varepsilon. \qquad (2.7)\end{aligned}$$

Again using (1.4), (2.1) and (2.2) we get

$$\begin{aligned}\left\|y_n^{k-1}-v_n^{k-1}\right\| &= \left\|\left(1-\beta_n^{k-1}\right)(x_n-u_n)+\beta_n^{k-1}\left(Tx_n-\widetilde{T}u_n\right)\right\| \\ &\leq \left(1-\beta_n^{k-1}\right)\|x_n-u_n\|+\beta_n^{k-1}\left\|Tx_n-\widetilde{T}u_n\right\| \\ &\leq \left(1-\beta_n^{k-1}\right)\|x_n-u_n\|+\beta_n^{k-1}\|Tx_n-Tu_n\| \\ &\quad +\beta_n^{k-1}\left\|Tu_n-\widetilde{T}u_n\right\| \\ &\leq \left[1-\beta_n^{k-1}(1-\delta)\right]\|x_n-u_n\|+\beta_n^{k-1}\varphi(\|x_n-Tx_n\|)+\beta_n^{k-1}\varepsilon. \qquad (2.8)\end{aligned}$$

Substituting (2.8) in (2.7) we have

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| &\leq [1-\alpha_n(1-\delta)] \\ &\quad \left[1-\beta_n^1(1-\delta)\right]\cdots\left[1-\beta_n^{k-1}(1-\delta)\right]\|x_n-u_n\| \\ &\quad +[1-\alpha_n(1-\delta)] \\ &\quad \left[1-\beta_n^1(1-\delta)\right]\cdots\left[1-\beta_n^{k-2}(1-\delta)\right]\beta_n^{k-1}\varphi(\|x_n-Tx_n\|) \\ &\quad +[1-\alpha_n(1-\delta)] \\ &\quad \left[1-\beta_n^1(1-\delta)\right]\cdots\left[1-\beta_n^{k-3}(1-\delta)\right]\beta_n^{k-2}\varphi\left(\left\|y_n^{k-1}-Ty_n^{k-1}\right\|\right) \\ &\quad +\cdots+[1-\alpha_n(1-\delta)]\,\beta_n^1\varphi\left(\left\|y_n^2-Ty_n^2\right\|\right)+\alpha_n\varphi\left(\left\|y_n^1-Ty_n^1\right\|\right) \\ &\quad +[1-\alpha_n(1-\delta)]\left[1-\beta_n^1(1-\delta)\right]\cdots\left[1-\beta_n^{k-2}(1-\delta)\right]\beta_n^{k-1}\varepsilon \\ &\quad +\cdots+[1-\alpha_n(1-\delta)]\,\beta_n^1\varepsilon+\alpha_n\varepsilon. \qquad (2.9)\end{aligned}$$

Since $\delta\in[0,1)$ and $\{\alpha_n\}_{n=0}^{\infty}$,$\left\{\beta_n^i\right\}_{n=0}^{\infty}\subset[0,1)$ for $i=\overline{1,k-1}$, we have

$$[1-\alpha_n(1-\delta)]\left[1-\beta_n^1(1-\delta)\right]\cdots\left[1-\beta_n^i(1-\delta)\right]\leq[1-\alpha_n(1-\delta)]\,. \qquad (2.10)$$

Using inequality (2.10) and assumption (i) in (2.9) it follows

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| &\leq [1-\alpha_n(1-\delta)]\|x_n-u_n\| \\ &+\alpha_n\varphi(\|x_n-Tx_n\|)+\alpha_n\varphi\left(\left\|y_n^{k-1}-Ty_n^{k-1}\right\|\right) \\ &+\cdots+\alpha_n\varphi\left(\left\|y_n^2-Ty_n^2\right\|\right)+\alpha_n\varphi\left(\left\|y_n^1-Ty_n^1\right\|\right) \\ &+\alpha_n\varepsilon+\alpha_n\varepsilon+\cdots+\alpha_n\varepsilon+\alpha_n\varepsilon \\ &= [1-\alpha_n(1-\delta)]\|x_n-u_n\| \\ &+\alpha_n(1-\delta)\left\{\frac{\varphi(\|x_n-Tx_n\|)+\varphi\left(\left\|y_n^{k-1}-Ty_n^{k-1}\right\|\right)}{1-\delta}\right. \\ &\left.+\cdots+\frac{\varphi\left(\left\|y_n^1-Ty_n^1\right\|\right)+k\varepsilon}{1-\delta}\right\}. \end{aligned} \tag{2.11}$$

Define

$$\begin{aligned} a_n &:= \|x_n-u_n\|, \\ \mu_n &:= \alpha_n(1-\delta)\in(0.1), \\ \eta_n &:= \left\{\frac{\varphi(\|x_n-Tx_n\|)+\varphi\left(\left\|y_n^{k-1}-Ty_n^{k-1}\right\|\right)}{1-\delta}\right. \\ &\left.+\cdots+\frac{\varphi\left(\left\|y_n^1-Ty_n^1\right\|\right)+k\varepsilon}{1-\delta}\right\}. \end{aligned}$$

From Theorem 1 it follows that $\lim_{n\to\infty}\|x_n-p\|=0$. Since T satisfies condition (1.4) and $Tp=p\in F_T$,

$$\begin{aligned} 0 &\leq \|x_n-Tx_n\| \\ &\leq \|x_n-p\|+\|Tp-Tx_n\| \\ &\leq \|x_n-p\|+\delta\|p-x_n\|+\varphi(\|p-Tp\|) \\ &= (1+\delta)\|x_n-p\|\to 0 \text{ as } n\to\infty. \end{aligned} \tag{2.12}$$

Since $\beta_n^i\in[0,1)$, $\forall n\in\mathbb{N}$, $i=\overline{1,k-1}$ and using (1.4) and (1.10) we have

$$\begin{aligned} 0 &\leq \left\|y_n^1-Ty_n^1\right\|=\left\|y_n^1-p+p-Ty_n^1\right\| \\ &\leq \left\|y_n^1-p\right\|+\left\|Tp-Ty_n^1\right\| \\ &\leq \left\|y_n^1-p\right\|+\delta\left\|p-y_n^1\right\|+\varphi(\|p-Tp\|) \\ &= (1+\delta)\left\|y_n^1-p\right\| \\ &= (1+\delta)\left\|\left(1-\beta_n^1\right)y_n^2+\beta_n^1Ty_n^2-p\left(1-\beta_n^1+\beta_n^1\right)\right\| \\ &\leq (1+\delta)\left\{\left(1-\beta_n^1\right)\left\|y_n^2-p\right\|+\beta_n^1\left\|Ty_n^2-Tp\right\|\right\} \\ &\leq (1+\delta)\left\{\left(1-\beta_n^1\right)\left\|y_n^2-p\right\|+\beta_n^1\delta\left\|y_n^2-p\right\|\right\} \\ &= (1+\delta)\left[1-\beta_n^1(1-\delta)\right]\left\|y_n^2-p\right\| \\ &\leq \cdots \\ &\leq (1+\delta)\left[1-\beta_n^1(1-\delta)\right]\cdots\left[1-\beta_n^{k-2}(1-\delta)\right]\left\|y_n^{k-1}-p\right\| \\ &\leq (1+\delta)\left[1-\beta_n^1(1-\delta)\right]\cdots\left[1-\beta_n^{k-1}(1-\delta)\right]\|x_n-p\| \\ &\leq (1+\delta)\|x_n-p\|\to 0 \text{ as } n\to\infty. \end{aligned} \tag{2.13}$$

It is easy to see from (2.13) that this result is also valid for $\left\|Ty_n^2 - y_n^2\right\|, \ldots, \left\|Ty_n^{k-1} - y_n^{k-1}\right\|$.

Since φ is continuous, we have

$$\begin{aligned} & \lim_{n\to\infty} \varphi\left(\|x_n - Tx_n\|\right) \\ = & \lim_{n\to\infty} \varphi\left(\left\|y_n^1 - Ty_n^1\right\|\right) = \cdots = \lim_{n\to\infty} \varphi\left(\left\|y_n^{k-1} - Ty_n^{k-1}\right\|\right) = 0. \end{aligned} \tag{2.14}$$

Hence an application of Lemma 1 to (2.11) leads to

$$\|p - q\| \leq \frac{k\varepsilon}{1-\delta}. \tag{2.15}$$

□

As shown by Hussain et al. ([21], Theorem 8), in an arbitrary Banach space X, the S-iteration $\{x_n\}_{n=0}^{\infty}$ given by (1.6) converges to the fixed point of T, where $T : E \to E$ is a mapping satisfying condition (1.3).

Theorem 3. *Let $T : E \to E$ be a map satisfying* (1.4) *with $F_T \neq \emptyset$ and $\{x_n\}_{n=0}^{\infty}$ be defined by* (1.6) *with real sequences $\{\beta_n\}_{n=0}^{\infty}$, $\{\alpha_n\}_{n=0}^{\infty} \subset [0, 1)$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges to the unique fixed point of T.*

Proof. The argument is similar to the proof of Theorem 8 of [21], and is thus omitted. □

We now prove result on data dependence for the S-iterative procedure by utilizing Theorem 3.

Theorem 4. *Let T, $\widetilde{T}$ be two operators as in Theorem 2. Let $\{x_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$ be S-iterations defined by (1.6) and with real sequences $\{\beta_n\}_{n=0}^{\infty}$, $\{\alpha_n\}_{n=0}^{\infty} \subset [0, 1)$ satisfying (i) $\frac{1}{2} \leq \alpha_n$, $\forall n \in \mathbb{N}$, and (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$. If $p = Tp$ and $q = \widetilde{T}q$, then we have*

$$\|p - q\| \leq \frac{3\varepsilon}{1-\delta}.$$

Proof. For a given $x_0 \in C$ and $u_0 \in C$ we consider following iteration for T and $\widetilde{T}$:

$$\begin{cases} x_0 \in E, \\ x_{n+1} = (1-\alpha_n) Tx_n + \alpha_n Ty_n, \\ y_n = (1-\beta_n) x_n + \beta_n Tx_n, \ n \in \mathbb{N} \end{cases} \tag{2.16}$$

and

$$\begin{cases} x_0 \in E, \\ u_{n+1} = (1-\alpha_n) \widetilde{T}u_n + \alpha_n \widetilde{T}v_n, \\ v_n = (1-\beta_n) u_n + \beta_n \widetilde{T}u_n, \ n \in \mathbb{N}. \end{cases} \tag{2.17}$$

Using (1.4), (2.16) and (2.17), we obtain the following estimates

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| &= \left\|(1-\alpha_n)\left(Tx_n-\widetilde{T}u_n\right)+\alpha_n\left(Ty_n-\widetilde{T}v_n\right)\right\| \\ &\le (1-\alpha_n)\left\|Tx_n-\widetilde{T}u_n\right\|+\alpha_n\left\|Ty_n-\widetilde{T}v_n\right\| \\ &= (1-\alpha_n)\left\|Tx_n-Tu_n+Tu_n-\widetilde{T}u_n\right\| \\ &\quad +\alpha_n\left\|Ty_n-Tv_n+Tv_n-\widetilde{T}v_n\right\| \\ &\le (1-\alpha_n)\left\{\|Tx_n-Tu_n\|+\left\|Tu_n-\widetilde{T}u_n\right\|\right\} \\ &\quad +\alpha_n\left\{\|Ty_n-Tv_n\|+\left\|Tv_n-\widetilde{T}v_n\right\|\right\} \\ &\le (1-\alpha_n)\{\delta\|x_n-u_n\|+\varphi(\|x_n-Tx_n\|)+\varepsilon\} \\ &\quad +\alpha_n\{\delta\|y_n-v_n\|+\varphi(\|y_n-Ty_n\|)+\varepsilon\}, \end{aligned} \tag{2.18}$$

$$\begin{aligned}\|y_n-v_n\| &= \left\|(1-\beta_n)(x_n-u_n)+\beta_n\left(Tx_n-\widetilde{T}u_n\right)\right\| \\ &\le (1-\beta_n)\|x_n-u_n\|+\beta_n\left\|Tx_n-\widetilde{T}u_n\right\| \\ &= (1-\beta_n)\|x_n-u_n\|+\beta_n\left\|Tx_n-Tu_n+Tu_n-\widetilde{T}u_n\right\| \\ &\le (1-\beta_n)\|x_n-u_n\|+\beta_n\left\{\|Tx_n-Tu_n\|+\left\|Tu_n-\widetilde{T}u_n\right\|\right\} \\ &\le (1-\beta_n)\|x_n-u_n\|+\beta_n\{\delta\|x_n-u_n\|+\varphi(\|x_n-Tx_n\|)+\varepsilon\} \\ &= [1-\beta_n(1-\delta)]\|x_n-u_n\|+\beta_n\varphi(\|x_n-Tx_n\|)+\beta_n\varepsilon. \end{aligned} \tag{2.19}$$

Combining (2.18) and (2.19),

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| &\le \{(1-\alpha_n)\delta+\alpha_n\delta[1-\beta_n(1-\delta)]\}\|x_n-u_n\| \\ &\quad +\{1-\alpha_n+\alpha_n\delta\beta_n\}\varphi(\|x_n-Tx_n\|)+\alpha_n\varphi(\|y_n-Ty_n\|) \\ &\quad +\alpha_n\delta\beta_n\varepsilon+(1-\alpha_n)\varepsilon+\alpha_n\varepsilon. \end{aligned} \tag{2.20}$$

For $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}\subset[0,1)$ and $\delta\in[0,1)$

$$(1-\alpha_n)\delta<1-\alpha_n,\ 1-\beta_n(1-\delta)<1,\ \alpha_n\delta\beta_n<\alpha_n. \tag{2.21}$$

It follows from assumption (i) that

$$1-\alpha_n<\alpha_n, \forall n\in\mathbb{N}. \tag{2.22}$$

Therefore, combining (2.22) and (2.21) to (2.20) gives

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| &\le [1-\alpha_n(1-\delta)]\|x_n-u_n\| \\ &\quad +2\alpha_n\varphi(\|x_n-Tx_n\|)+\alpha_n\varphi(\|y_n-Ty_n\|) \\ &\quad +\alpha_n\varepsilon+\alpha_n\varepsilon+\alpha_n\varepsilon, \end{aligned} \tag{2.23}$$

or, equivalently,

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| &\le [1-\alpha_n(1-\delta)]\|x_n-u_n\| \\ &\quad +\alpha_n(1-\delta)\frac{\{2\varphi(\|x_n-Tx_n\|)+\varphi(\|y_n-Ty_n\|)+3\varepsilon\}}{1-\delta}. \end{aligned} \tag{2.24}$$

Now define

$$\begin{aligned} a_n &: = \|x_n - u_n\|, \\ \eta_n &: = \alpha_n(1-\delta) \in (0,1) \\ \rho_n &: = \frac{2\varphi(\|x_n - Tx_n\|) + \varphi(\|y_n - Ty_n\|) + 3\varepsilon}{1-\delta}. \end{aligned}$$

From Theorem 3, we have $\lim_{n\to\infty}\|x_n - p\| = 0$. Since T satisfies condition (1.4), and $Tp = p \in F_T$, using an argument similar to that in the proof of Theorem 2

$$\lim_{n\to\infty}\|x_n - Tx_n\| = \lim_{n\to\infty}\|y_n - Ty_n\| = 0. \tag{2.25}$$

Using the fact that φ is continuous we have

$$\lim_{n\to\infty}\varphi(\|x_n - Tx_n\|) = \lim_{n\to\infty}\varphi(\|y_n - Ty_n\|) = 0. \tag{2.26}$$

An application of Lemma 1 to (2.24) leads to

$$\|p - q\| \le \frac{3\varepsilon}{1-\delta}. \tag{2.27}$$

□

3. Conclusion

Since the iterative schemes (1.7) and (1.8) are special cases the iterative process (1.10), Theorem 1 generalizes Theorem 2.1 of [24], and Theorem 2.1 of [15]. By taking $k = 3$ and $k = 2$ in Theorem 2, data dependence results for the iterative schemes (1.7) and (1.8) can be easily obtained. For $k = 3$, Theorem 2 reduces to Theorem 3.2 of [25]. Since condition (1.4) is more general than condition (1.3), Theorem 3 generalizes Theorem 8 of [21].

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